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AN ANALYTIC SOLUTION OF THE PROBLEM OF THE TEMPERATURE JUMPS AND VAPOUR DENSITY OVER A SURFACE WHEN THERE IS A TEMPERATURE GRADIENT†

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An analytic solution of the Boltzmann equation with a BGK (Bhatnagar, Gross and Krook) collision operator is constructed in the problem of the temperature jumps and the density of a rarefied gas in the half-space above a volatile surface, where a constant temperature gradient is specified far from the surface. The necessary numerical calculations are carried out. A canonical-matrix method with a normal form at infinity is simultaneously developed to solve the Riemann–Hilbert vector boundary-value problem. The proof of the expansion of the solution of the boundary-value problem considered in generalized eigenvectors of the corresponding characteristic equation is reduced to the solution of this problem.

This problem has been solved by different approximate methods in many publications (for example, [1–8]). The history of the problem is described in [8–10], and also in the other publications mentioned.

Below, we obtain an accurate solution (in closed form) of this classical problem, expressed in quadratures, and we carry out numerical calculations using the exact formulae. The Boltzmann equation is reduced to a vector integro-differential equation with symmetric kernel, which is solved by Case's method, which consists of expanding the solution in generalized eigenvectors. The proof of the theorem of the expansion is equivalent to the solution of the Riemann–Hilbert vector boundary-value problem with a matrix coefficient, the diagonalizing matrix of which is analytic in the plane with branch cuts connecting branching points. Hence, to construct the factor-matrix for the coefficient it is necessary to solve two other additional matrix boundary-value problems on the branch cuts. Along the way a canonical-matrix method is developed for solving the boundary-value problem.

The fundamental-matrix method was developed previously [11] to solve the corresponding boundary-value problem. In this paper a canonical matrix is used. Note that although the canonical matrix itself (in the temperature-jump problem) was constructed in [12], it was not used to solve the corresponding boundary-value problem.

The equations considered and their analogues are widely employed not only in kinetic theory but also in theoretical astrophysics, in plasma physics, and in neutron-transport theory (for more detail see, for example, [13–15]).

Suppose a rarefied monatomic gas occupies the half-space $x > 0$ above a volatile surface which lies in the $x = 0$ plane. A steady temperature field $T(x) = T_0(1 + Kx)$ is maintained in the gas far from the surface. The behaviour of the gas is described by the distribution function f, which is the solution of the Boltzmann equation

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$$\upsilon_x \frac{\partial}{\partial x} f(x,\mathbf{C}) = Lf$$

and satisfies the boundary conditions

$$f(x,\mathbf{C}) = f^{(0)}\left[1 + (Kx - AC_x + \varepsilon_T)(C^2 - 5/2) + \varepsilon_n + \varepsilon_T\right] \ (x \to \infty)$$

$$f(0,\mathbf{C}) = f^{(0)}, \ C_x > 0$$

Here

$$\varepsilon_T = (T_0 - T_w)/T_w, \ \varepsilon_n = (n_0 - n_w)/n_w$$

$$f^{(0)} = n_w(\beta_w/\pi)^{3/2}\exp(-C^2)$$

$$\mathbf{C} = \beta_w^{1/2}\mathbf{v}, \ \beta_w = m/(2kT_w), \ A = 3Kl/\sqrt{\pi}$$

(ε_T and ε_n are the initial values of the temperature jumps and the density, respectively, T_w is the temperature of the wall, and n_w is the saturated-vapour density).

The novelty of this problem from the physical point of view, compared with that considered in [11], is the fact that in this problem the gas occupies a half-space above the volatile (permeable) surface, through which there is no mass flux from the surface, i.e. the gas is in mechanical equilibrium with the surface. The presence of a temperature gradient in the system denotes thermal non-equilibrium. This means that the concentration of the gas in the region of the surface is different from the equilibrium value—the concentration of the saturated vapour at the surface temperature.

The main problems from the physical point of view considered in this paper are to calculate the relative deviation of the vapour concentration from the equilibrium value (the value of ε_n) and to calculate the relative temperature jump (the value of ε_T).

We will seek a solution of the Boltzmann equation in the form

$$f = f^{(0)}[1 + (Kx - AC_x)(C^2 - 5/2) + Y(x,\mathbf{C})]$$

and we will expand Y in two orthogonal directions

$$Y = Y_1(x,C_x) + (C_y^2 + C_z^2 - 1)Y_2(x,C_x)$$

Then, we obtain the following equation for the column vector

$$\left(\mu\frac{\partial}{\partial x} + 1\right)Y(x,\mu) = \frac{1}{\sqrt{\pi}} Q(\mu)\int_{-\infty}^{\infty} Q^T(\mu')Y(x,\mu')\exp(-\mu'^2)d\mu' \tag{1}$$

$$\left(Q(\mu) = \begin{Vmatrix} \gamma\left(\mu^2 - \frac{1}{2}\right) & 1 \\ \gamma & 0 \end{Vmatrix}, \ \gamma^2 = \frac{2}{3}\right)$$

with boundary conditions

$$Y(0,\mu) = A\mu\begin{Vmatrix} \mu^2 - \frac{3}{2} \\ 1 \end{Vmatrix} \ (\mu > 0)$$

$$Y(\infty,\mu)=\varepsilon_T\left\|\begin{matrix}\mu^2-\dfrac{1}{2}\\ 1\end{matrix}\right\|+\varepsilon_n\left\|\begin{matrix}1\\ 0\end{matrix}\right\| \quad (\mu<0) \tag{2}$$

(the superscript T denotes transposition).

We seek a solution of Eq. (1) in the form

$$Y_\eta(x,\mu)=\exp(-x/\eta)F(\eta,\mu) \tag{3}$$

and we arrive at the characteristic equation

$$(\eta-\mu)F(\eta,\mu)=\frac{1}{\sqrt{\pi}}\eta Q(\mu)n(\eta)$$
$$\left(n(\eta)=\int_{-\infty}^{\infty}\exp(-\mu^2)Q^T(\mu)F(\eta,\mu)d\mu\right) \tag{4}$$

where $n(\eta)$ is a non-singular normalizing vector. Hence we obtain the following eigenvectors of the characteristic equation

$$F(\eta,\mu)=\left[\frac{1}{\sqrt{\pi}}\eta Q(\mu)P\frac{1}{\eta-\mu}+\exp(\eta^2)Q^{-T}(\eta)\Lambda(\eta)\delta(\eta-\mu)\right]n(\eta)$$
$$\left(\Lambda(z)=I+\frac{1}{\sqrt{\pi}}z\int_{-\infty}^{\infty}Q^T(\mu)Q(\mu)\exp(-\mu^2)\frac{d\mu}{\mu-z}\right) \tag{5}$$

The symbol Px^{-1} denotes a distribution—the principal value of the integral of x^{-1}, $\delta(x)$ is the delta function $\Lambda(z)$ is the dispersion matrix, I is the unit matrix, and $Q^{-T}(\mu)$ is the inverse transposed matrix.

It can be shown that the dispersion equation

$$\lambda(z)\equiv\det\Lambda(z)=0$$

has a fourth-order zero at infinity, to which the following solutions of Eq. (1) correspond

$$Y_1=\left\|\begin{matrix}\mu^2-\frac{1}{2}\\ 1\end{matrix}\right\|,\ Y_2=\left\|\begin{matrix}1\\ 0\end{matrix}\right\|,\ Y_3=(\mu-x)Y_1,\ Y_4=(\mu-x)Y_2 \tag{6}$$

Theorem. Equation (1) with boundary conditions (2) has a unique solution, which can be represented in the form of the expansion

$$Y(x,\mu)=\varepsilon_T\left\|\begin{matrix}\mu^2-\frac{1}{2}\\ 1\end{matrix}\right\|+\varepsilon_n\left\|\begin{matrix}1\\ 0\end{matrix}\right\|+\int_0^{\infty}\exp\left(-\frac{x}{\eta}\right)F(\eta,\mu)d\eta \tag{7}$$

Proof. We substitute $x=0$ and the eigenvectors (5) into (7). We obtain a singular vector integral equation with a Cauchy kernel on the half-axis $\mu>0$

$$\psi(\mu)=\frac{1}{\sqrt{\pi}}Q(\mu)\int_0^{\infty}\frac{\eta n(\eta)}{\eta-\mu}d\eta+\exp(\mu^2)Q^{-T}(\mu)\Lambda(\mu)n(\mu)$$
$$\left(\psi(\mu)=A\mu\left\|\begin{matrix}\mu^2-\dfrac{3}{2}\\ 1\end{matrix}\right\|-\varepsilon_T\left\|\begin{matrix}\mu^2-\dfrac{1}{2}\\ 1\end{matrix}\right\|-\varepsilon_n\left\|\begin{matrix}1\\ 0\end{matrix}\right\|\right) \tag{8}$$

Multiplying (8) on the left by $\mu\exp(-\mu^2)Q^T(\mu)$ and using the boundary values on the half-axis of the dispersion matrix and the vector

$$N(z)=\frac{1}{2\pi i}\int_0^{\infty}\frac{\eta n(\eta)}{\eta-z}\,d\eta \tag{9}$$

we obtain a homogeneous Riemann–Hilbert vector boundary-value problem with matrix coefficient, which, after transposition, can be written in the form

$$[2\sqrt{\pi}iN^+(\mu)-Q^{-1}(\mu)\psi(\mu)]^T\Lambda^+(\mu)=[2\sqrt{\pi}iN^-(\mu)-Q^{-1}(\mu)\psi(\mu)]^T\Lambda^-(\mu),\quad \mu>0 \tag{10}$$

Multiplying (10) on the right by $Q^{-1}(\mu)Q^{-T}(\mu)$ and putting

$$W(z)=\Lambda(z)Q^{-1}(z)Q^{-T}(z) \tag{11}$$

we obtain the following boundary-value problem

$$[2\sqrt{\pi}iN^+(\mu)-Q^{-1}(\mu)\psi(\mu)]^T W^+(\mu)=[2\sqrt{\pi}iN^-(\mu)-Q^{-1}(\mu)\psi(\mu)]^T W^-(\mu),\quad \mu>0 \tag{12}$$

Note that the matrix

$$S(z)=\begin{Vmatrix} 1/\gamma & \frac{1}{2}[r(z)-z^2-\frac{1}{2}] \\ -1/\gamma & \frac{1}{2}[r(z)+z^2+\frac{1}{2}] \end{Vmatrix}$$

$$(r(z)=\sqrt{w(z)},\quad w(z)=z^4-3z^2+25/4)$$

reduces the matrix $W(z)$ to diagonal form

$$S(z)W(z)S^{-1}(z)=\operatorname{diag}\{\Omega_1(z),\ \Omega_2(z)\}\equiv\Omega(z)$$

Here

$$\Omega_\alpha(z)=1/4\left[11/2-z^2+(-1)^{\alpha-1}r(z)+4zt(z)\right]\quad(\alpha=1,2)$$

are the elements of the diagonal matrix $\Omega(z)$.

The matrix-function $S(z)$ is unique and analytic in the plane with branch cuts Γ_1 and Γ_2, connecting the branching points $-\bar{a}$ and a, and $-a$ and $\bar{a}$, respectively, which are zeros of the polynomial $w(z)(a)=\sqrt{(2)}+i/\sqrt{(2)})$.

For the matrix coefficient

$$G(\mu)=\Lambda^+(\mu)\left[\Lambda^-(\mu)\right]^{-1} \tag{13}$$

that occurs in the boundary condition (10), we consider the factorization problem

$$\Phi^+(\mu)=G(\mu)\Phi^-(\mu),\quad \mu>0 \tag{14}$$

Moreover, for the matrix-function $\Phi(z)$ to be unique it is necessary for the following condition to be satisfied on the branch-cut $\Gamma=\Gamma_1\cup\Gamma_2$

$$\Phi^+(\tau)=\Phi^-(\tau),\quad \tau\in\Gamma \tag{15}$$

We will seek a solution of problems (15) and (14) in the form

$$\Phi(z) = S^{-1}(z)U(z)S(z) \tag{16}$$

where $U(z)$ is a new unknown matrix.

Boundary-value problems (14) and (15) are now equivalent to the following boundary-value problems

$$U^{+}(\mu) = G_0(\mu)U^{-}(\mu), \quad \mu > 0 \tag{17}$$

$$U^{+}(\tau)T = TU^{-}(\tau), \quad \tau \in \Gamma \tag{18}$$

Here

$$G_0(\mu) = S(\mu)G(\mu)S^{-1}(\mu) = \Omega^{+}(\mu)\left[\Omega^{-}(\mu)\right]^{-1}$$

$$T = -S^{+}(\tau)\left[S^{-}(\tau)\right]^{-1} = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}$$

Since the matrix G_0 is diagonal, it is convenient to take U also in the form of a diagonal matrix: $\operatorname{diag}\{U_1, U_2\} \equiv U$. The matrix boundary-value problem (17) can now be split into two scalar boundary-value problems

$$U_\alpha^{+}(\mu) = \frac{\Omega_\alpha^{+}(\mu)}{\Omega_\alpha^{-}(\mu)}\, U_\alpha^{-}(\mu) \;\; (\alpha = 1,2), \;\; \mu > 0 \tag{19}$$

while problem (18) essentially remains a vector boundary-value problem

$$U_1^{\pm}(\tau) = U_2^{\mp}(\tau), \quad \tau \in \Gamma \tag{20}$$

Without deriving it, we will give the solution of problems (19) and (20)

$$U_\alpha(z) = (z - x_1)U_\alpha^{(0)}(z) \;\; (\alpha = 1,2) \tag{21}$$

$$U_\alpha^{(0)}(z) = \exp\left\{A(z) + (-1)^{\alpha-1} r(z)(B(z) - R(z))\right\}$$

$$A(z) = \frac{1}{2\pi}\int_0^\infty \frac{d(x)}{x - z}\, dx, \quad B(z) = \frac{1}{2\pi}\int_0^\infty \frac{b(x)dx}{r(x)(x - z)}$$

$$R(z) = \int_0^{x_1} \frac{dx}{r(x)(x - z)}$$

$$a(x) = \theta_1(x) + \theta_2(x) - 2\pi, \quad b(x) = \theta_2(x) - \theta_1(x)$$

The point x_1 is the solution of the Jacobi inversion problem

$$\frac{1}{2\pi}\int_0^\infty \frac{b(x)}{r(x)}\, dx = \int_0^{x_1} \frac{dx}{r(x)}$$

Here $\theta_\alpha(x)$ is the principal value of the argument of the function $\Omega_\alpha^{+}(x)$.

Thus, the factor-matrix $\Phi(z)$ for the dispersion matrix $\Lambda(z)$ is constructed and defined by expression (16). Another factorization of the dispersion matrix from [16] is now necessary

$$\Lambda(z) = \Phi_0(z)\Phi_0^{T}(-z) \tag{22}$$

where $\Phi_0(z)$ is a canonical matrix with normal form at infinity.

We substitute (22) into (10), multiply it on the right by $\Phi_0^{T}(-\mu)$ and apply a transposition to the equation obtained. We obtain the boundary-value problem

$$[\Phi_0^+(\mu)]^T[2\sqrt{\pi}iN^+(\mu)-Q^{-1}(\mu)\psi(\mu)]=[\Phi_0^-(\mu)]^T[2\sqrt{\pi}iN^-(\mu)-Q^{-1}(\mu)\psi(\mu)],\ \mu>0 \qquad (23)$$

Taking into account the behaviour at infinity of the matrices and vectors in this boundary condition (23), we can write its general solution

$$2\sqrt{\pi}iN(z)=(Az-\varepsilon_T)\left\| \begin{matrix} 1/\gamma \\ -1 \end{matrix} \right\| -(\varepsilon_n+\varepsilon_T)\left\| \begin{matrix} 0 \\ 1 \end{matrix} \right\| +\Phi_0^{-T}(z)\left\| \begin{matrix} c_1 \\ c_2 \end{matrix} \right\| \qquad (24)$$

where c_1 and c_2 are arbitrary constants. We obtain the matrix $\Phi_0^{-T}(z)$ from (60) and (61) of [12]

$$\Phi_0^{-T}(z)=\sqrt{12/5}\,\Phi^{-T}(z)\Xi(-z) \qquad (25)$$

where the matrix $\Xi(z)$ is defined by (62) in [12].

We will make the solution (24) correct, i.e. we will determine the unknown constants ε_T, ε_n, c_1 and c_2 so that this solution will decrease as $1/z$ at infinity. This vector can then be taken as the vector $N(z)$ given by (9).

Suppose p_n and $q_n (n=0, -1, -2, \dots)$ are Laurent coefficients of the expansions of the functions $[U_1^{(0)}(z)]^{-1}$ and $[U_2^{(0)}(z)]^{-1}$ respectively in the neighbourhood of infinity, where, by (21), $p_0q_0=1$. We will denote by A_{-n}, B_{-n} and R_{-n} $(n=1, 2, \dots)$ the Laurent coefficients of the expansions in the neighbourhood of infinity of the functions $A(z), B(z)$ and $R(z)$

$$A_{-n}=\frac{1}{2\pi}\int_0^\infty a(x)x^{n-1}dx,\quad B_{-n}=-\frac{1}{2\pi}\int_0^\infty\frac{b(x)}{r(x)}\,x^{n-1}dx$$
$$R_{-n}=-\int_0^{x_1}\frac{x^{n-1}}{r(x)}\,dx \qquad (26)$$

Equating the coefficients of z and z^0 in the upper and lower rows of (23) to zero, we obtain a system of equations, from which we have

$$c_1=-\frac{\sqrt{5}}{2}A,\quad c_2=0$$
$$\varepsilon_T=A[-x_1-p_0p_{-1}+\sqrt{3}\,b_{12}q_0] \qquad (27)$$
$$\varepsilon_n=A[x_1+p_0p_{-1}-(q_0-2p_0)\sqrt{3}\,b_{12}]$$

where the coefficient b_{12} is defined in [12]. The free parameters of the vector $N(z)$ are defined uniquely, including also the coefficients ε_T and ε_n of expansion (7), corresponding to a discrete spectrum. The coefficients of the continuous spectrum $n(\mu)$ are also found uniquely from Sokhotskii's formula: $N^+(\mu)-N^-(\mu)=\mu n(\mu)$. This proves the theorem. The last two relations of (27) give the required values of the temperature jumps and the gas density above the volatile surface. We will write these formulae in explicit form

$$\varepsilon_T=-\frac{3Kl}{\sqrt{\pi}}\{2x_1Q-A_{-1}+B_{-3}-R_{-1}\}$$

$$\varepsilon_n=-\varepsilon_T-\frac{12Kl}{\sqrt{\pi}}x_1Q\exp\{-2A_{-2}+2R_{-2}\} \qquad (28)$$

$$Q=(\gamma^2+\beta\gamma x_1-\alpha^2)\ (\gamma-\alpha)^{-2}$$

$$\alpha=-\frac{1}{\sqrt{6}}\left[r(x_1)+x_1^2+\frac{1}{2}\right],\ \beta=-\gamma x_1\left(1+\frac{x_1^2-3/2}{r(x_1)}\right)$$

(the coefficients A_{-1}, A_{-2}, R_{-1}, R_{-2}, B_{-3} are defined by (26)).

Numerical calculations, carried out using the accurate formulae (28), give the following values $\varepsilon_T=2.15897\,Kl$ and $\varepsilon_n=1.23035\,Kl$ for the temperature jump and the density jump in the rarefied gas, respectively. Up to the present time the exact value of the temperature jump was assumed to be $\varepsilon_T=2.1646984\,Kl$ [16].

The first formula of (28) obtained here is identical with the formula (30) in [11].

The proposed method can be used to solve the similar Riemann–Hilbert boundary-value problem with matrix coefficient (the diagonalizing matrix of which has a branching point), that occurs in the theory of the scattering of polarized light (see, for example, [17]).

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